

ON NEUTROSOPHIC BIDIRECTED GRAPHS AND THEIR STRUCTURAL PROPERTIES

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ABSTRACT. In this paper, we introduce the concept of neutrosophic bidirected graphs as a generalization of intuitionistic fuzzy bidirected graphs by incorporating truth-membership, indeterminacy-membership, and falsity-membership functions on vertices and edges together with a bidirection mapping. Several fundamental properties and structural results are established. In particular, we investigate reductions to fuzzy and crisp bidirected graphs, characterize neutrosophic directed graphs induced by bidirections, and study operations such as intersection and union. Furthermore, strong neutrosophic bidirected graphs, homomorphisms, and isomorphisms are discussed. The proposed structure provides a useful mathematical model for handling uncertainty and indeterminacy in bidirected network structures.

1. INTRODUCTION

Graph theory has become an essential branch of mathematics with wide-ranging applications in computer science, communication systems, artificial intelligence, transportation networks, optimization, and decision sciences. Classical graph models, however, are often inadequate for representing uncertainty, inconsistency, and incomplete information that naturally arise in practical systems. To overcome these difficulties, several generalized graph models based on fuzzy and neutrosophic frameworks have been developed.

The theory of fuzzy sets, introduced by Zadeh [1], provides an effective mathematical tool for handling uncertainty through membership functions. Rosenfeld [2] initiated the concept of fuzzy graphs by extending fuzzy set theory to graph structures. Later, Atanassov [3] generalized fuzzy sets to intuitionistic fuzzy sets by introducing both membership and non-membership degrees. Since then, intuitionistic fuzzy graphs and their algebraic and structural properties have been studied extensively by many researchers [4, 5, 6].

In recent years, neutrosophic set theory introduced by Smarandache [7] has attracted considerable attention because of its ability to characterize uncertainty, indeterminacy, and inconsistency simultaneously through truth-membership, indeterminacy-membership, and falsity-membership functions. Compared with intuitionistic fuzzy theory, neutrosophic theory offers a more flexible framework for modeling incomplete and inconsistent information. The development of neutrosophic graph theory has therefore become an active area of research. Several important contributions related to neutrosophic graphs, neutrosophic digraphs, and their applications can be found in [8, 9, 10, 11, 12].

Bidirected graphs constitute another important generalization of graph structures in which each edge possesses local orientations at its end vertices. These graphs are useful in network flow analysis, communication systems, transportation models, electrical circuit analysis, and

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combinatorial optimization. Fuzzy and intuitionistic fuzzy bidirected graph models were introduced to incorporate uncertainty into bidirectional network structures. However, those models cannot explicitly represent indeterminate information.

Motivated by the above observations, in this paper we introduce the concept of neutrosophic bidirected graphs as a natural extension of intuitionistic fuzzy bidirected graphs. The proposed framework combines neutrosophic information with bidirectional graph structures by assigning truth-membership, indeterminacy-membership, and falsity-membership values to vertices and edges together with a bidirection mapping. Several fundamental properties and structural characterizations are established. In particular, we investigate reductions to fuzzy and crisp bidirected graphs, induced neutrosophic directed graphs, intersection and union operations, strong neutrosophic bidirected graphs, homomorphisms, and isomorphisms.

The obtained results provide a generalized mathematical setting for modeling uncertainty and indeterminacy in bidirected network systems and may serve as a foundation for further studies in neutrosophic graph theory and its applications in intelligent systems, data analysis, optimization, and decision-making problems.

2. FUNDAMENTAL CONCEPTS

Definition 2.1 ([13]). *A neutrosophic set $\tilde{A}$ on $\tilde{X}$ is characterized by three functions*

$$\top_{\tilde{A}} : \tilde{X} \rightarrow [0, 1], \quad \text{I}_{\tilde{A}} : \tilde{X} \rightarrow [0, 1], \quad \text{and} \quad \text{F}_{\tilde{A}} : \tilde{X} \rightarrow [0, 1],$$

called the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively, such that

$$0 \leq \top_{\tilde{A}}(\partial) + \text{I}_{\tilde{A}}(\partial) + \text{F}_{\tilde{A}}(\partial) \leq 3$$

for all $\partial \in \tilde{X}$.

Definition 2.2. *Let $\mathcal{G}^* = (\mathcal{V}, \text{E})$ be a crisp undirected simple graph. A neutrosophic graph (NG) on $\mathcal{G}^*$ is defined as*

$$\mathcal{G} = ((\top, \text{I}, \text{F}), (\top_{\text{E}}, \text{I}_{\text{E}}, \text{F}_{\text{E}})),$$

where

$$\top, \text{I}, \text{F} : \mathcal{V} \rightarrow [0, 1],$$

and

$$\top_{\text{E}}, \text{I}_{\text{E}}, \text{F}_{\text{E}} : \text{E} \rightarrow [0, 1],$$

are the truth-membership, indeterminacy-membership, and falsity-membership functions on vertices and edges, respectively, satisfying

$$0 \leq \top(\partial) + \text{I}(\partial) + \text{F}(\partial) \leq 3,$$

$$0 \leq \top_{\text{E}}(\{\zeta, \partial\}) + \text{I}_{\text{E}}(\{\zeta, \partial\}) + \text{F}_{\text{E}}(\{\zeta, \partial\}) \leq 3,$$

and

$$\top_{\text{E}}(\{\zeta, \partial\}) \leq \min\{\top(\zeta), \top(\partial)\},$$

$$\text{I}_{\text{E}}(\{\zeta, \partial\}) \leq \max\{\text{I}(\zeta), \text{I}(\partial)\},$$

$$\text{F}_{\text{E}}(\{\zeta, \partial\}) \geq \max\{\text{F}(\zeta), \text{F}(\partial)\},$$

for all $\{\zeta, \partial\} \in \text{E}$.

Example 2.1. *Consider the crisp undirected simple graph $\mathcal{G}^* = (\mathcal{V}, \text{E})$, where $\mathcal{V} = \{\partial_1, \partial_2, \partial_3, \partial_4\}$ and $\text{E} = \{\{\partial_1, \partial_2\}, \{\partial_1, \partial_3\}, \{\partial_2, \partial_3\}, \{\partial_3, \partial_4\}\}$. Define the neutrosophic vertex functions by*

$$\top(\partial_1) = 0.9, \quad \text{I}(\partial_1) = 0.1, \quad \text{F}(\partial_1) = 0.1,$$

$$\top(\partial_2) = 0.7, \quad \text{I}(\partial_2) = 0.3, \quad \text{F}(\partial_2) = 0.2,$$

$$\top(\partial_3) = 0.6, \quad \text{I}(\partial_3) = 0.4, \quad \text{F}(\partial_3) = 0.3,$$

$$\top(\partial_4) = 0.5, \quad \text{I}(\partial_4) = 0.5, \quad \text{F}(\partial_4) = 0.4.$$

Clearly,

$$0 \leq \top(\partial_i) + \mathbf{l}(\partial_i) + \mathbf{F}(\partial_i) \leq 3$$

for each $i = 1, 2, 3, 4$, since

$$0.9 + 0.1 + 0.1 = 1.1,$$

$$0.7 + 0.3 + 0.2 = 1.2,$$

$$0.6 + 0.4 + 0.3 = 1.3,$$

and

$$0.5 + 0.5 + 0.4 = 1.4.$$

Now, we define the neutrosophic edge functions as follows:

$$\top_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.7, \quad \mathbf{l}_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.3, \quad \mathbf{F}_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.2,$$

$$\top_{\mathbf{E}}(\{\partial_1, \partial_3\}) = 0.6, \quad \mathbf{l}_{\mathbf{E}}(\{\partial_1, \partial_3\}) = 0.4, \quad \mathbf{F}_{\mathbf{E}}(\{\partial_1, \partial_3\}) = 0.3,$$

$$\top_{\mathbf{E}}(\{\partial_2, \partial_3\}) = 0.5, \quad \mathbf{l}_{\mathbf{E}}(\{\partial_2, \partial_3\}) = 0.4, \quad \mathbf{F}_{\mathbf{E}}(\{\partial_2, \partial_3\}) = 0.3,$$

$$\top_{\mathbf{E}}(\{\partial_3, \partial_4\}) = 0.5, \quad \mathbf{l}_{\mathbf{E}}(\{\partial_3, \partial_4\}) = 0.5, \quad \mathbf{F}_{\mathbf{E}}(\{\partial_3, \partial_4\}) = 0.4.$$

We verify the defining conditions.

For the edge $\{\partial_1, \partial_2\}$,

$$\top_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.7 \leq \min\{0.9, 0.7\} = 0.7,$$

$$\mathbf{l}_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.3 \leq \max\{0.1, 0.3\} = 0.3,$$

and

$$\mathbf{F}_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.2 \geq \max\{0.1, 0.2\} = 0.2.$$

For the edge $\{\partial_1, \partial_3\}$,

$$0.6 \leq \min\{0.9, 0.6\} = 0.6,$$

$$0.4 \leq \max\{0.1, 0.4\} = 0.4,$$

and

$$0.3 \geq \max\{0.1, 0.3\} = 0.3.$$

For the edge $\{\partial_2, \partial_3\}$,

$$0.5 \leq \min\{0.7, 0.6\} = 0.6,$$

$$0.4 \leq \max\{0.3, 0.4\} = 0.4,$$

and

$$0.3 \geq \max\{0.2, 0.3\} = 0.3.$$

For the edge $\{\partial_3, \partial_4\}$,

$$0.5 \leq \min\{0.6, 0.5\} = 0.5,$$

$$0.5 \leq \max\{0.4, 0.5\} = 0.5,$$

and

$$0.4 \geq \max\{0.3, 0.4\} = 0.4.$$

Moreover,

$$0 \leq \top_{\mathbf{E}}(\hat{\mathbf{g}}) + \mathbf{l}_{\mathbf{E}}(\hat{\mathbf{g}}) + \mathbf{F}_{\mathbf{E}}(\hat{\mathbf{g}}) \leq 3$$

for every edge $\hat{\mathbf{g}} \in \mathbf{E}$. Hence, $\mathcal{G} = ((\top, \mathbf{l}, \mathbf{F}), (\top_{\mathbf{E}}, \mathbf{l}_{\mathbf{E}}, \mathbf{F}_{\mathbf{E}}))$ is a neutrosophic graph on $\mathcal{G}^*$.

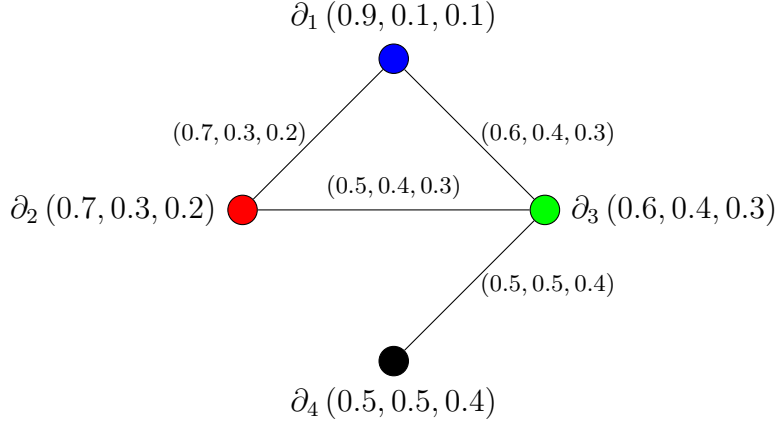


FIGURE 1. Neutrosophic graph corresponding to Example 2.1

Definition 2.3. Let $\vec{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{A}})$ be a crisp directed simple graph. A neutrosophic directed graph (NDG) on $\vec{\mathcal{G}}$ is defined as $\vec{\mathcal{G}}_N = ((\top, \mathbf{l}, \mathbf{F}), (\top_{\tilde{\mathcal{A}}}, \mathbf{l}_{\tilde{\mathcal{A}}}, \mathbf{F}_{\tilde{\mathcal{A}}}))$, where

$$\top, \mathbf{l}, \mathbf{F} : \mathcal{V} \rightarrow [0, 1], \text{ and } \top_{\tilde{\mathcal{A}}}, \mathbf{l}_{\tilde{\mathcal{A}}}, \mathbf{F}_{\tilde{\mathcal{A}}} : \tilde{\mathcal{A}} \rightarrow [0, 1],$$

are the truth-membership, indeterminacy-membership, and falsity-membership functions on vertices and arcs, respectively, satisfying

$$0 \leq \top(\partial) + \mathbf{l}(\partial) + \mathbf{F}(\partial) \leq 3,$$

$$0 \leq \top_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) + \mathbf{l}_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) + \mathbf{F}_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) \leq 3,$$

and

$$\top_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) \leq \min\{\top(\iota_1), \top(\iota_2)\},$$

$$\mathbf{l}_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) \leq \max\{\mathbf{l}(\iota_1), \mathbf{l}(\iota_2)\},$$

$$\mathbf{F}_{\tilde{\mathcal{A}}}(\iota_1, \iota_2) \geq \max\{\mathbf{F}(\iota_1), \mathbf{F}(\iota_2)\},$$

for all $(\iota_1, \iota_2) \in \tilde{\mathcal{A}}$.

Example 2.2. Consider the crisp directed graph $\vec{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{A}})$, where $\mathcal{V} = \{\partial_1, \partial_2, \partial_3\}$ and $\tilde{\mathcal{A}} = \{(\partial_1, \partial_2), (\partial_2, \partial_3), (\partial_1, \partial_3)\}$. Define the neutrosophic vertex functions

$$\top(\partial_1) = 0.8, \quad \mathbf{l}(\partial_1) = 0.2, \quad \mathbf{F}(\partial_1) = 0.1,$$

$$\top(\partial_2) = 0.6, \quad \mathbf{l}(\partial_2) = 0.3, \quad \mathbf{F}(\partial_2) = 0.2,$$

$$\top(\partial_3) = 0.7, \quad \mathbf{l}(\partial_3) = 0.4, \quad \mathbf{F}(\partial_3) = 0.2.$$

Clearly,

$$0 \leq \top(\partial_i) + \mathbf{l}(\partial_i) + \mathbf{F}(\partial_i) \leq 3$$

for each $i = 1, 2, 3$, since

$$0.8 + 0.2 + 0.1 = 1.1,$$

$$0.6 + 0.3 + 0.2 = 1.1,$$

and

$$0.7 + 0.4 + 0.2 = 1.3.$$

Now define the neutrosophic arc functions as follows:

$$\top_{\tilde{\mathcal{A}}}(\partial_1, \partial_2) = 0.5, \quad \mathbf{l}_{\tilde{\mathcal{A}}}(\partial_1, \partial_2) = 0.3, \quad \mathbf{F}_{\tilde{\mathcal{A}}}(\partial_1, \partial_2) = 0.2,$$

$$\top_{\tilde{\mathcal{A}}}(\partial_2, \partial_3) = 0.6, \quad \mathbf{l}_{\tilde{\mathcal{A}}}(\partial_2, \partial_3) = 0.4, \quad \mathbf{F}_{\tilde{\mathcal{A}}}(\partial_2, \partial_3) = 0.2,$$

$$\top_{\tilde{\mathcal{A}}}(\partial_1, \partial_3) = 0.7, \quad \mathbf{l}_{\tilde{\mathcal{A}}}(\partial_1, \partial_3) = 0.4, \quad \mathbf{F}_{\tilde{\mathcal{A}}}(\partial_1, \partial_3) = 0.2.$$

We verify the required conditions.

For the arc (∂_1, ∂_2) ,

$$\begin{aligned}\top_{\tilde{A}}(\partial_1, \partial_2) &= 0.5 \leq \min\{0.8, 0.6\} = 0.6, \\ \text{I}_{\tilde{A}}(\partial_1, \partial_2) &= 0.3 \leq \max\{0.2, 0.3\} = 0.3,\end{aligned}$$

and

$$\text{F}_{\tilde{A}}(\partial_1, \partial_2) = 0.2 \geq \max\{0.1, 0.2\} = 0.2.$$

Similarly, for the arc (∂_2, ∂_3) ,

$$0.6 \leq \min\{0.6, 0.7\} = 0.6,$$

$$0.4 \leq \max\{0.3, 0.4\} = 0.4,$$

and

$$0.2 \geq \max\{0.2, 0.2\} = 0.2.$$

For the arc (∂_1, ∂_3) ,

$$0.7 \leq \min\{0.8, 0.7\} = 0.7,$$

$$0.4 \leq \max\{0.2, 0.4\} = 0.4,$$

and

$$0.2 \geq \max\{0.1, 0.2\} = 0.2.$$

Moreover,

$$0 \leq \top_{\tilde{A}}(\iota_1, \iota_2) + \text{I}_{\tilde{A}}(\iota_1, \iota_2) + \text{F}_{\tilde{A}}(\iota_1, \iota_2) \leq 3$$

for every arc $(\iota_1, \iota_2) \in \tilde{\mathbf{A}}$. Hence, $\vec{\mathcal{G}}_N = ((\top, \text{I}, \text{F}), (\top_{\tilde{A}}, \text{I}_{\tilde{A}}, \text{F}_{\tilde{A}}))$ is a neutrosophic directed graph.

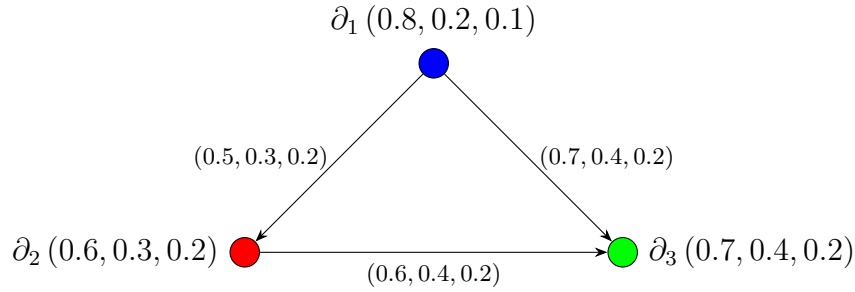


FIGURE 2. Neutrosophic directed graph corresponding to Example 2.2

3. MAIN RESULTS

Definition 3.1. Let $\mathcal{G}^* = (\mathcal{V}, \mathbf{E})$ be a crisp undirected simple graph, where $\mathcal{V}$ is a nonempty set of vertices and $\mathbf{E} \subseteq \{\{\zeta, \partial\} : \zeta, \partial \in \mathcal{V}, \zeta \neq \partial\}$ is its edge set.

A neutrosophic bidirected graph is a septuple $\mathcal{B}_N = (\top, \text{I}, \text{F}, \top_{\mathbf{E}}, \text{I}_{\mathbf{E}}, \text{F}_{\mathbf{E}}, \xi)$, where

$$\top, \text{I}, \text{F} : \mathcal{V} \rightarrow [0, 1], \quad \top_{\mathbf{E}}, \text{I}_{\mathbf{E}}, \text{F}_{\mathbf{E}} : \mathbf{E} \rightarrow [0, 1], \quad \text{and } \xi : \mathcal{V} \times \mathbf{E} \rightarrow \{-1, 0, 1\},$$

satisfying the following conditions:

(1) For every $\partial \in \mathcal{V}$,

$$0 \leq \top(\partial) + \text{I}(\partial) + \text{F}(\partial) \leq 3. \quad (\text{Vertex neutrosophic constraint})$$

(2) For every $\hat{\mathbf{g}} \in \mathbf{E}$,

$$0 \leq \top_{\mathbf{E}}(\hat{\mathbf{g}}) + \text{I}_{\mathbf{E}}(\hat{\mathbf{g}}) + \text{F}_{\mathbf{E}}(\hat{\mathbf{g}}) \leq 3. \quad (\text{Edge neutrosophic constraint})$$

(3) For every $\hat{\mathbf{g}} = \{\zeta, \partial\} \in \mathbf{E}$,

$$\top_{\mathbf{E}}(\hat{\mathbf{g}}) \leq \min\{\top(\zeta), \top(\partial)\},$$

$$\text{I}_{\mathbf{E}}(\hat{\mathbf{g}}) \leq \max\{\text{I}(\zeta), \text{I}(\partial)\},$$

$$\text{F}_{\mathbf{E}}(\hat{\mathbf{g}}) \geq \max\{\text{F}(\zeta), \text{F}(\partial)\}. \quad (\text{Membership consistency})$$

(4)

$$\xi(\partial, \hat{\mathbf{g}}) = 0 \iff \partial \notin \hat{\mathbf{g}}. \text{ (Incidence condition)}$$

(5) For every $\hat{\mathbf{g}} = \{\zeta, \partial\} \in \mathbf{E}$,

$$\xi(\zeta, \hat{\mathbf{g}}), \xi(\partial, \hat{\mathbf{g}}) \in \{-1, 1\}. \text{ (Local orientation)}$$

Example 3.1. Consider the crisp undirected simple graph $\mathcal{G}^* = (\mathcal{V}, \mathbf{E})$, where $V = \{\partial_1, \partial_2, \partial_3\}$ and $\mathbf{E} = \{\{\partial_1, \partial_2\}, \{\partial_2, \partial_3\}, \{\partial_1, \partial_3\}\}$. We define the neutrosophic vertex functions by

$$\begin{aligned} \top(\partial_1) &= 0.8, \quad \text{I}(\partial_1) = 0.2, \quad \text{F}(\partial_1) = 0.1, \\ \top(\partial_2) &= 0.6, \quad \text{I}(\partial_2) = 0.3, \quad \text{F}(\partial_2) = 0.2, \\ \top(\partial_3) &= 0.7, \quad \text{I}(\partial_3) = 0.4, \quad \text{F}(\partial_3) = 0.2. \end{aligned}$$

Clearly,

$$0 \leq \top(\partial_i) + \text{I}(\partial_i) + \text{F}(\partial_i) \leq 3$$

for each $i = 1, 2, 3$, since

$$\top(\partial_1) + \text{I}(\partial_1) + \text{F}(\partial_1) = 1.1, \quad \top(\partial_2) + \text{I}(\partial_2) + \text{F}(\partial_2) = 1.1, \quad \text{and} \quad \top(\partial_3) + \text{I}(\partial_3) + \text{F}(\partial_3) = 1.3.$$

Now, we define the neutrosophic edge functions as follows:

$$\begin{aligned} \top_{\mathbf{E}}(\{\partial_1, \partial_2\}) &= 0.5, \quad \text{I}_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.3, \quad \text{F}_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.2, \\ \top_{\mathbf{E}}(\{\partial_2, \partial_3\}) &= 0.6, \quad \text{I}_{\mathbf{E}}(\{\partial_2, \partial_3\}) = 0.4, \quad \text{F}_{\mathbf{E}}(\{\partial_2, \partial_3\}) = 0.2, \\ \top_{\mathbf{E}}(\{\partial_1, \partial_3\}) &= 0.7, \quad \text{I}_{\mathbf{E}}(\{\partial_1, \partial_3\}) = 0.4, \quad \text{F}_{\mathbf{E}}(\{\partial_1, \partial_3\}) = 0.2. \end{aligned}$$

Next, we now define the bidirection function

$$\xi : \mathcal{V} \times \mathbf{E} \rightarrow \{-1, 0, 1\}$$

by

$$\begin{aligned} \xi(\partial_1, \{\partial_1, \partial_2\}) &= 1, & \xi(\partial_2, \{\partial_1, \partial_2\}) &= -1, \\ \xi(\partial_2, \{\partial_2, \partial_3\}) &= 1, & \xi(\partial_3, \{\partial_2, \partial_3\}) &= -1, \\ \xi(\partial_1, \{\partial_1, \partial_3\}) &= -1, & \xi(\partial_3, \{\partial_1, \partial_3\}) &= 1. \end{aligned}$$

Also, $\xi(\partial, \hat{\mathbf{g}}) = 0$ whenever $\partial \notin \hat{\mathbf{g}}$.

Now, we verify the required conditions. For the edge $\{\partial_1, \partial_2\}$,

$$\begin{aligned} \top_{\mathbf{E}}(\{\partial_1, \partial_2\}) &= 0.5 \leq \min\{0.8, 0.6\} = 0.6, \\ \text{I}_{\mathbf{E}}(\{\partial_1, \partial_2\}) &= 0.3 \leq \max\{0.2, 0.3\} = 0.3, \end{aligned}$$

and

$$\text{F}_{\mathbf{E}}(\{\partial_1, \partial_2\}) = 0.2 \geq \max\{0.1, 0.2\} = 0.2.$$

Similarly, for the edge $\{\partial_2, \partial_3\}$,

$$\begin{aligned} 0.6 &\leq \min\{0.6, 0.7\} = 0.6, \\ 0.4 &\leq \max\{0.3, 0.4\} = 0.4, \end{aligned}$$

and

$$0.2 \geq \max\{0.2, 0.2\} = 0.2.$$

For the edge $\{\partial_1, \partial_3\}$,

$$\begin{aligned} 0.7 &\leq \min\{0.8, 0.7\} = 0.7, \\ 0.4 &\leq \max\{0.2, 0.4\} = 0.4, \end{aligned}$$

and

$$0.2 \geq \max\{0.1, 0.2\} = 0.2.$$

Moreover,

$$0 \leq \top_{\mathbf{E}}(\hat{\mathbf{g}}) + \text{I}_{\mathbf{E}}(\hat{\mathbf{g}}) + \text{F}_{\mathbf{E}}(\hat{\mathbf{g}}) \leq 3$$

for every $\hat{\mathbf{g}} \in \mathbf{E}$.

The incidence condition holds because

$$\xi(\partial, \hat{g}) = 0 \iff \partial \notin \hat{g}.$$

Further, for every edge

$$\hat{g} = \{\zeta, \partial\} \in E,$$

we have

$$\xi(\zeta, \hat{g}), \xi(\partial, \hat{g}) \in \{-1, 1\},$$

which satisfies the local orientation condition. Hence, $\mathcal{B}_N = (\top, \text{I}, \text{F}, \top_E, \text{I}_E, \text{F}_E, \xi)$ is a neutrosophic bidirected graph.

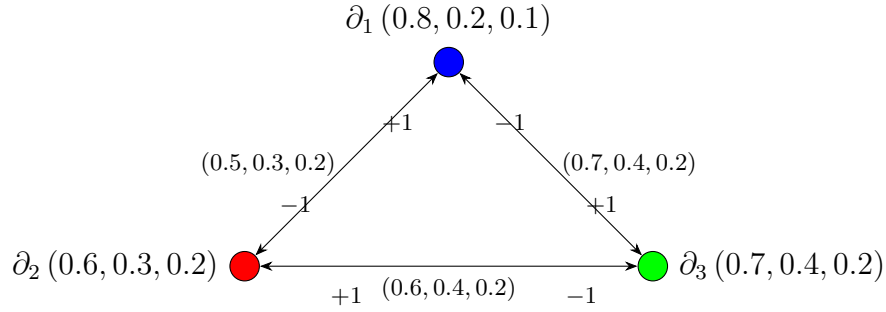


FIGURE 3. Neutrosophic bidirected graph corresponding to Example 3.1

Example 3.2. Consider the crisp undirected simple graph $\mathcal{G}^* = (\mathcal{V}, E)$, where $\mathcal{V} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and

$$E = \{\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}, \{\alpha_3, \alpha_4\}, \{\alpha_1, \alpha_4\}\}.$$

Now, we define the neutrosophic vertex functions by

$$\begin{aligned} \top(\alpha_1) &= 0.9, \quad \text{I}(\alpha_1) = 0.1, \quad \text{F}(\alpha_1) = 0.1, \\ \top(\alpha_2) &= 0.7, \quad \text{I}(\alpha_2) = 0.2, \quad \text{F}(\alpha_2) = 0.2, \\ \top(\alpha_3) &= 0.6, \quad \text{I}(\alpha_3) = 0.3, \quad \text{F}(\alpha_3) = 0.2, \\ \top(\alpha_4) &= 0.8, \quad \text{I}(\alpha_4) = 0.2, \quad \text{F}(\alpha_4) = 0.1. \end{aligned}$$

Next, define the neutrosophic edge functions by

$$\begin{aligned} \top_E(\{\alpha_1, \alpha_2\}) &= 0.7, \quad \text{I}_E(\{\alpha_1, \alpha_2\}) = 0.2, \quad \text{F}_E(\{\alpha_1, \alpha_2\}) = 0.2, \\ \top_E(\{\alpha_2, \alpha_3\}) &= 0.6, \quad \text{I}_E(\{\alpha_2, \alpha_3\}) = 0.3, \quad \text{F}_E(\{\alpha_2, \alpha_3\}) = 0.2, \\ \top_E(\{\alpha_3, \alpha_4\}) &= 0.6, \quad \text{I}_E(\{\alpha_3, \alpha_4\}) = 0.3, \quad \text{F}_E(\{\alpha_3, \alpha_4\}) = 0.2, \\ \top_E(\{\alpha_1, \alpha_4\}) &= 0.8, \quad \text{I}_E(\{\alpha_1, \alpha_4\}) = 0.2, \quad \text{F}_E(\{\alpha_1, \alpha_4\}) = 0.1. \end{aligned}$$

Further, define the bidirection function

$$\xi : \mathcal{V} \times E \rightarrow \{-1, 0, 1\}$$

by

$$\begin{aligned} \xi(\alpha_1, \{\alpha_1, \alpha_2\}) &= 1, & \xi(\alpha_2, \{\alpha_1, \alpha_2\}) &= -1, \\ \xi(\alpha_2, \{\alpha_2, \alpha_3\}) &= 1, & \xi(\alpha_3, \{\alpha_2, \alpha_3\}) &= -1, \\ \xi(\alpha_3, \{\alpha_3, \alpha_4\}) &= 1, & \xi(\alpha_4, \{\alpha_3, \alpha_4\}) &= -1, \\ \xi(\alpha_1, \{\alpha_1, \alpha_4\}) &= -1, & \xi(\alpha_4, \{\alpha_1, \alpha_4\}) &= 1. \end{aligned}$$

Hence, $\mathcal{B}_N = (\top, \text{I}, \text{F}, \top_E, \text{I}_E, \text{F}_E, \xi)$ is a neutrosophic bidirected graph.

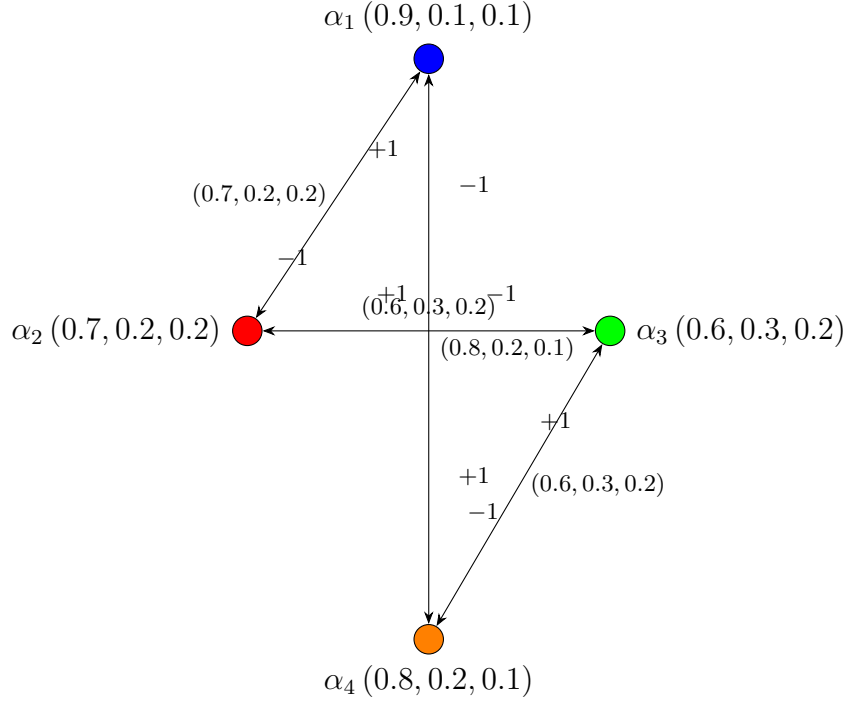


FIGURE 4. Neutrosophic bidirected graph corresponding to the above example 3.2

Theorem 3.1. Let $\mathcal{B}_N = (\top, \mathbf{l}, \mathbf{F}, \top_E, \mathbf{l}_E, \mathbf{F}_E, \xi)$ be a neutrosophic bidirected graph on $\mathcal{G}^* = (\mathcal{V}, \mathbf{E})$. Then:

- (1) If $\mathbf{l}(\partial) = 0$, $\mathbf{F}(\partial) = 0$ for all $\partial \in \mathcal{G}$, and $\mathbf{l}_E(\hat{\mathbf{g}}) = 0$, $\mathbf{F}_E(\hat{\mathbf{g}}) = 0$ for all $\hat{\mathbf{g}} \in \mathbf{E}$, then $\mathcal{B}_N$ reduces to a fuzzy bidirected graph.
- (2) If, in addition, $\top(\partial) = 1$ for all $\partial \in \mathcal{V}$, and $\top_E(\hat{\mathbf{g}}) = 1$ for all $\hat{\mathbf{g}} \in \mathbf{E}$, then $\mathcal{B}_N$ further reduces to a crisp bidirected graph.
- (3) If $\xi(\zeta, \hat{\mathbf{g}}) = -\xi(\partial, \hat{\mathbf{g}})$ for every edge $\hat{\mathbf{g}} = \{\zeta, \partial\} \in \mathbf{E}$. Then, $\mathcal{B}_N$ induces a neutrosophic directed graph on the digraph

$$\tilde{\mathbf{A}} = \left\{ (\iota_1, \iota_2) \mid \{\iota_1, \iota_2\} \in \mathbf{E}, \xi(\iota_1, \{\iota_1, \iota_2\}) = +1, \xi(\iota_2, \{\iota_1, \iota_2\}) = -1 \right\}.$$

Proof. (1) Setting $\mathbf{l} \equiv 0$, $\mathbf{F} \equiv 0$, and $\mathbf{l}_E \equiv 0$, $\mathbf{F}_E \equiv 0$ in $\mathcal{B}_N$ leaves exactly the data $(\top, \top_E, \xi)$, satisfying the axioms of a fuzzy bidirected graph.

(2) Further imposing $\top \equiv 1$, $\top_E \equiv 1$ makes all membership consistency conditions trivial, recovering the usual crisp bidirected graph $(\mathcal{G}^*, \xi)$.

(3) Assume $\xi(\zeta, \hat{\mathbf{g}}) = -\xi(\partial, \hat{\mathbf{g}})$ for every edge $\hat{\mathbf{g}} = \{\zeta, \partial\} \in \mathbf{E}$.

we define

$$\tilde{\mathbf{A}} = \left\{ (\iota_1, \iota_2) \in \mathcal{V} \times \mathcal{V} : \{\iota_1, \iota_2\} \in \mathbf{E}, \xi(\iota_1, \{\iota_1, \iota_2\}) = +1, \xi(\iota_2, \{\iota_1, \iota_2\}) = -1 \right\}.$$

Then, $\vec{\mathcal{G}} = (\mathcal{V}, \tilde{\mathbf{A}})$ is a crisp directed graph. Define

$$\top_{\tilde{\mathbf{A}}}(\iota_1, \iota_2) = \top_E(\{\iota_1, \iota_2\}),$$

$$\mathbf{l}_{\tilde{\mathbf{A}}}(\iota_1, \iota_2) = \mathbf{l}_E(\{\iota_1, \iota_2\}),$$

and

$$\mathbf{F}_{\tilde{\mathbf{A}}}(\iota_1, \iota_2) = \mathbf{F}_E(\{\iota_1, \iota_2\}),$$

and retain the vertex functions $\top$, $\mathbf{l}$, $\mathbf{F}$. Since

$$\top_E(\hat{\mathbf{g}}) \leq \min\{\top(\zeta), \top(\partial)\},$$

$$\mathbf{l}_E(\hat{\mathbf{g}}) \leq \max\{\mathbf{l}(\zeta), \mathbf{l}(\partial)\},$$

and

$$F_E(\hat{g}) \geq \max\{F(\zeta), F(\partial)\}$$

hold in $\mathcal{B}_N$, it follows that

$$\begin{aligned} \top_{\tilde{A}}(\iota_1, \iota_2) &\leq \min\{\top(\iota_1), \top(\iota_2)\}, \\ \mathbf{l}_{\tilde{A}}(\iota_1, \iota_2) &\leq \max\{\mathbf{l}(\iota_1), \mathbf{l}(\iota_2)\}, \end{aligned}$$

and

$$F_{\tilde{A}}(\iota_1, \iota_2) \geq \max\{F(\iota_1), F(\iota_2)\}.$$

Moreover,

$$0 \leq \top_{\tilde{A}}(\iota_1, \iota_2) + \mathbf{l}_{\tilde{A}}(\iota_1, \iota_2) + F_{\tilde{A}}(\iota_1, \iota_2) \leq 3.$$

Hence,

$$(\top, \mathbf{l}, F, \top_{\tilde{A}}, \mathbf{l}_{\tilde{A}}, F_{\tilde{A}})$$

satisfies the axioms of a neutrosophic directed graph on $\vec{\mathcal{G}}$. $\square$

Theorem 3.2. *Let $\mathcal{B}_{N_1} = (\top_1, \mathbf{l}_1, F_1, \top_{E_1}, \mathbf{l}_{E_1}, F_{E_1}, \xi)$ and $\mathcal{B}_{N_2} = (\top_2, \mathbf{l}_2, F_2, \top_{E_2}, \mathbf{l}_{E_2}, F_{E_2}, \xi)$ be two neutrosophic bidirected graphs on the same crisp graph $\mathcal{G}^* = (\mathcal{V}, E)$. Define $\top = \top_1 \cap \top_2$, $\mathbf{l} = \mathbf{l}_1 \cup \mathbf{l}_2$, and $F = F_1 \cup F_2$, and similarly for edge functions. Then, $(\top, \mathbf{l}, F, \top_E, \mathbf{l}_E, F_E, \xi)$ is also a neutrosophic bidirected graph on $\mathcal{G}^*$.*

Proof. We verify each axiom.

1. Vertex neutrosophic constraint. Fix $\partial \in \mathcal{V}$. Since

$$0 \leq \top_i(\partial) + \mathbf{l}_i(\partial) + F_i(\partial) \leq 3 \quad \text{for } i = 1, 2,$$

we obtain

$$\top(\partial) + \mathbf{l}(\partial) + F(\partial) = \min\{\top_1(\partial), \top_2(\partial)\} + \max\{\mathbf{l}_1(\partial), \mathbf{l}_2(\partial)\} + \max\{F_1(\partial), F_2(\partial)\}.$$

Clearly,

$$0 \leq \top(\partial) + \mathbf{l}(\partial) + F(\partial) \leq 3.$$

2. Edge neutrosophic constraint. Similarly, for every $\hat{g} \in E$,

$$0 \leq \top_E(\hat{g}) + \mathbf{l}_E(\hat{g}) + F_E(\hat{g}) \leq 3.$$

3. Membership consistency. For any edge

$$\hat{g} = \{\zeta, \partial\} \in E,$$

we have

$$\top_E(\hat{g}) = \min\{\top_{E_1}(\hat{g}), \top_{E_2}(\hat{g})\}.$$

Since

$$\top_{E_i}(\hat{g}) \leq \min\{\top_i(\zeta), \top_i(\partial)\}$$

for $i = 1, 2$, it follows that

$$\top_E(\hat{g}) \leq \min\{\top(\zeta), \top(\partial)\}.$$

Also,

$$\mathbf{l}_E(\hat{g}) = \max\{\mathbf{l}_{E_1}(\hat{g}), \mathbf{l}_{E_2}(\hat{g})\}$$

and

$$\mathbf{l}_{E_i}(\hat{g}) \leq \max\{\mathbf{l}_i(\zeta), \mathbf{l}_i(\partial)\}.$$

Thus,

$$\mathbf{l}_E(\hat{g}) \leq \max\{\mathbf{l}(\zeta), \mathbf{l}(\partial)\}.$$

Likewise,

$$F_E(\hat{g}) = \max\{F_{E_1}(\hat{g}), F_{E_2}(\hat{g})\}$$

and

$$F_{E_i}(\hat{g}) \geq \max\{F_i(\zeta), F_i(\partial)\}.$$

Hence,

$$F_E(\hat{g}) \geq \max\{F(\zeta), F(\partial)\}.$$

4. Incidence condition. Since the bidirection function ξ is unchanged,

$$\xi(\partial, \hat{g}) = 0 \iff \partial \notin \hat{g}$$

still holds.

5. Local orientation. Similarly,

$$\xi(\zeta, \hat{g}), \xi(\partial, \hat{g}) \in \{-1, 1\}$$

for every edge

$$\hat{g} = \{\zeta, \partial\} \in E.$$

Therefore,

$$(\top, I, F, \top_E, I_E, F_E, \xi)$$

satisfies all axioms of a neutrosophic bidirected graph on $\mathcal{G}^*$. $\square$

Theorem 3.3 (Union of Neutrosophic Bidirected Graphs). *Let $\mathcal{B}_{N_1} = (\top_1, I_1, F_1, \top_{E_1}, I_{E_1}, F_{E_1}, \xi)$ and $\mathcal{B}_{N_2} = (\top_2, I_2, F_2, \top_{E_2}, I_{E_2}, F_{E_2}, \xi)$ be two neutrosophic bidirected graphs on the same underlying crisp graph $\mathcal{G}^* = (\mathcal{V}, E)$, sharing the same bidirection function ξ .*

Define

$$\top(\partial) = \max\{\top_1(\partial), \top_2(\partial)\},$$

$$I(\partial) = \min\{I_1(\partial), I_2(\partial)\},$$

$$F(\partial) = \min\{F_1(\partial), F_2(\partial)\},$$

and

$$\top_E(\hat{g}) = \max\{\top_{E_1}(\hat{g}), \top_{E_2}(\hat{g})\},$$

$$I_E(\hat{g}) = \min\{I_{E_1}(\hat{g}), I_{E_2}(\hat{g})\},$$

$$F_E(\hat{g}) = \min\{F_{E_1}(\hat{g}), F_{E_2}(\hat{g})\}.$$

Then, $(\top, I, F, \top_E, I_E, F_E, \xi)$ is again a neutrosophic bidirected graph on $\mathcal{G}^*$.

Proof. We verify the defining axioms.

1. Vertex neutrosophic constraint. For each $\partial \in \mathcal{V}$,

$$\top(\partial) + I(\partial) + F(\partial) = \max\{\top_1(\partial), \top_2(\partial)\} + \min\{I_1(\partial), I_2(\partial)\} + \min\{F_1(\partial), F_2(\partial)\}.$$

Since

$$0 \leq \top_i(\partial) + I_i(\partial) + F_i(\partial) \leq 3$$

for $i = 1, 2$, it follows immediately that

$$0 \leq \top(\partial) + I(\partial) + F(\partial) \leq 3.$$

2. Edge neutrosophic constraint. Similarly, for every $\hat{g} \in E$,

$$0 \leq \top_E(\hat{g}) + I_E(\hat{g}) + F_E(\hat{g}) \leq 3.$$

3. Membership consistency. Let

$$\hat{g} = \{\zeta, \partial\} \in E.$$

Since

$$\top_{E_i}(\hat{g}) \leq \min\{\top_i(\zeta), \top_i(\partial)\}$$

for $i = 1, 2$, we obtain

$$\top_E(\hat{g}) = \max\{\top_{E_1}(\hat{g}), \top_{E_2}(\hat{g})\} \leq \min\{\top(\zeta), \top(\partial)\}.$$

Also,

$$I_{E_i}(\hat{g}) \leq \max\{I_i(\zeta), I_i(\partial)\}.$$

Hence,

$$l_E(\hat{g}) = \min\{l_{E_1}(\hat{g}), l_{E_2}(\hat{g})\} \leq \max\{l(\zeta), l(\partial)\}.$$

Likewise,

$$F_{E_i}(\hat{g}) \geq \max\{F_i(\zeta), F_i(\partial)\},$$

therefore

$$F_E(\hat{g}) = \min\{F_{E_1}(\hat{g}), F_{E_2}(\hat{g})\} \geq \max\{F(\zeta), F(\partial)\}.$$

4. Incidence condition. Since the bidirection function ξ remains unchanged,

$$\xi(\partial, \hat{g}) = 0 \iff \partial \notin \hat{g}.$$

5. Local orientation. For every edge

$$\hat{g} = \{\zeta, \partial\} \in E,$$

we still have

$$\xi(\zeta, \hat{g}), \xi(\partial, \hat{g}) \in \{-1, 1\}.$$

Hence,

$$(\top, l, F, \top_E, l_E, F_E, \xi)$$

satisfies all axioms of a neutrosophic bidirected graph on $\mathcal{G}^*$. $\square$

Theorem 3.4. Let $\mathcal{B}_N = (\top, l, F, \top_E, l_E, F_E, \xi)$ be a neutrosophic bidirected graph on $\mathcal{G}^* = (\mathcal{V}, E)$. Define the complement $\mathcal{B}_N^c = (\top^c, l^c, F^c, \top_E^c, l_E^c, F_E^c, \xi)$, where for every $\partial \in \mathcal{V}$ and $\hat{g} \in E$, $\top^c(\partial) = F(\partial)$, $l^c(\partial) = l(\partial)$, $F^c(\partial) = \top(\partial)$, and $\top_E^c(\hat{g}) = F_E(\hat{g})$, $l_E^c(\hat{g}) = l_E(\hat{g})$, $F_E^c(\hat{g}) = \top_E(\hat{g})$. Then, $\mathcal{B}_N^c$ is also a neutrosophic bidirected graph.

Proof. For every $\partial \in \mathcal{V}$,

$$0 \leq \top(\partial) + l(\partial) + F(\partial) \leq 3.$$

Hence,

$$0 \leq \top^c(\partial) + l^c(\partial) + F^c(\partial) \leq 3.$$

Similarly,

$$0 \leq \top_E^c(\hat{g}) + l_E^c(\hat{g}) + F_E^c(\hat{g}) \leq 3.$$

Now,

$$\top_E^c(\hat{g}) = F_E(\hat{g}) \leq \min\{F(\zeta), F(\partial)\} = \min\{\top^c(\zeta), \top^c(\partial)\}.$$

Also,

$$l_E^c(\hat{g}) = l_E(\hat{g}) \leq \max\{l(\zeta), l(\partial)\} = \max\{l^c(\zeta), l^c(\partial)\}.$$

Likewise,

$$F_E^c(\hat{g}) = \top_E(\hat{g}) \geq \max\{\top(\zeta), \top(\partial)\} = \max\{F^c(\zeta), F^c(\partial)\}.$$

The incidence and orientation conditions remain unchanged since ξ is preserved. Therefore $\mathcal{B}_N^c$ is a neutrosophic bidirected graph. $\square$

Theorem 3.5. Let $\mathcal{B}_N = (\top, l, F, \top_E, l_E, F_E, \xi)$ be a neutrosophic bidirected graph. If for every edge $\hat{g} = \{\zeta, \partial\} \in E$, the equalities

$$\top_E(\hat{g}) = \min\{\top(\zeta), \top(\partial)\}, l_E(\hat{g}) = \max\{l(\zeta), l(\partial)\}, \text{ and } F_E(\hat{g}) = \max\{F(\zeta), F(\partial)\}$$

hold. Then, $\mathcal{B}_N$ is called a strong neutrosophic bidirected graph.

Proof. The result follows directly from the defining equalities, which strengthen the membership consistency conditions of a neutrosophic bidirected graph. $\square$

Theorem 3.6. Let $\mathcal{B}_{N_1} = (\top_1, l_1, F_1, \top_{E_1}, l_{E_1}, F_{E_1}, \xi_1)$ and $\mathcal{B}_{N_2} = (\top_2, l_2, F_2, \top_{E_2}, l_{E_2}, F_{E_2}, \xi_2)$ be neutrosophic bidirected graphs. Suppose $\phi : \partial_1 \rightarrow \partial_2$ is a graph homomorphism satisfying $\top_1(\partial) \leq \top_2(\phi(\partial))$, $l_1(\partial) \geq l_2(\phi(\partial))$, and $F_1(\partial) \geq F_2(\phi(\partial))$ for all $\partial \in \partial_1$. Then, ϕ preserves the neutrosophic bidirected structure.

Proof. Let $\hat{g} = \{\zeta, \partial\} \in E_1$. Since $\phi : \partial_1 \rightarrow \partial_2$ is a graph homomorphism, we have

$$\phi(\hat{g}) = \{\phi(\zeta), \phi(\partial)\} \in E_2.$$

Now, because $\mathcal{B}_{N_1}$ is a neutrosophic bidirected graph, the edge membership functions satisfy

$$\begin{aligned} \top_{E_1}(\hat{g}) &\leq \min\{\top_1(\zeta), \top_1(\partial)\}, \\ \mathbf{l}_{E_1}(\hat{g}) &\leq \max\{\mathbf{l}_1(\zeta), \mathbf{l}_1(\partial)\}, \end{aligned}$$

and

$$\mathbf{F}_{E_1}(\hat{g}) \geq \max\{\mathbf{F}_1(\zeta), \mathbf{F}_1(\partial)\}.$$

Using the assumptions

$$\begin{aligned} \top_1(\iota_1) &\leq \top_2(\phi(\iota_1)), \\ \mathbf{l}_1(\iota_1) &\geq \mathbf{l}_2(\phi(\iota_1)), \end{aligned}$$

and

$$\mathbf{F}_1(\iota_1) \geq \mathbf{F}_2(\phi(\iota_1))$$

for every $\iota_1 \in \partial_1$, we obtain

$$\min\{\top_1(\zeta), \top_1(\partial)\} \leq \min\{\top_2(\phi(\zeta)), \top_2(\phi(\partial))\}.$$

Hence,

$$\top_{E_1}(\hat{g}) \leq \min\{\top_2(\phi(\zeta)), \top_2(\phi(\partial))\}.$$

Similarly,

$$\max\{\mathbf{l}_1(\zeta), \mathbf{l}_1(\partial)\} \geq \max\{\mathbf{l}_2(\phi(\zeta)), \mathbf{l}_2(\phi(\partial))\}.$$

Therefore,

$$\mathbf{l}_{E_1}(\hat{g}) \leq \max\{\mathbf{l}_2(\phi(\zeta)), \mathbf{l}_2(\phi(\partial))\}.$$

Also,

$$\max\{\mathbf{F}_1(\zeta), \mathbf{F}_1(\partial)\} \geq \max\{\mathbf{F}_2(\phi(\zeta)), \mathbf{F}_2(\phi(\partial))\},$$

which implies

$$\mathbf{F}_{E_1}(\hat{g}) \geq \max\{\mathbf{F}_2(\phi(\zeta)), \mathbf{F}_2(\phi(\partial))\}.$$

Moreover, since $\mathcal{B}_{N_2}$ is a neutrosophic bidirected graph,

$$0 \leq \top_2(\phi(\partial)) + \mathbf{l}_2(\phi(\partial)) + \mathbf{F}_2(\phi(\partial)) \leq 3$$

for all $\partial \in \partial_1$, and

$$0 \leq \top_{E_2}(\phi(\hat{g})) + \mathbf{l}_{E_2}(\phi(\hat{g})) + \mathbf{F}_{E_2}(\phi(\hat{g})) \leq 3.$$

The bidirection and incidence conditions are preserved under ϕ , since adjacency and edge incidences are preserved by graph homomorphisms.

Therefore, ϕ preserves the neutrosophic bidirected structure from $\mathcal{B}_{N_1}$ to $\mathcal{B}_{N_2}$. $\square$

Theorem 3.7. Let $\mathcal{B}_{N_1} = (\top_1, \mathbf{l}_1, \mathbf{F}_1, \top_{E_1}, \mathbf{l}_{E_1}, \mathbf{F}_{E_1}, \xi_1)$ and $\mathcal{B}_{N_2} = (\top_2, \mathbf{l}_2, \mathbf{F}_2, \top_{E_2}, \mathbf{l}_{E_2}, \mathbf{F}_{E_2}, \xi_2)$ be two neutrosophic bidirected graphs. If $\phi : \partial_1 \rightarrow \partial_2$ preserves all vertex and edge neutrosophic values, then $\mathcal{B}_{N_1} \cong \mathcal{B}_{N_2}$.

Proof. Since ϕ is a graph isomorphism, it is bijective and preserves adjacency as well as incidences between vertices and edges.

Let $\hat{g} = \{\zeta, \partial\} \in E_1$. Then,

$$\phi(\hat{g}) = \{\phi(\zeta), \phi(\partial)\} \in E_2.$$

Using the assumptions,

$$\top_{E_1}(\hat{g}) = \top_{E_2}(\phi(\hat{g})),$$

and

$$\top_1(\zeta) = \top_2(\phi(\zeta)), \quad \top_1(\partial) = \top_2(\phi(\partial)).$$

Since $\mathcal{B}_{N_1}$ satisfies

$$\top_{E_1}(\hat{g}) \leq \min\{\top_1(\zeta), \top_1(\partial)\},$$

we obtain

$$\top_{E_2}(\phi(\hat{g})) \leq \min\{\top_2(\phi(\zeta)), \top_2(\phi(\partial))\}.$$

Similarly,

$$l_{E_2}(\phi(\hat{g})) \leq \max\{l_2(\phi(\zeta)), l_2(\phi(\partial))\},$$

and

$$F_{E_2}(\phi(\hat{g})) \geq \max\{F_2(\phi(\zeta)), F_2(\phi(\partial))\}.$$

Thus, all neutrosophic consistency conditions are preserved under ϕ .

Moreover, because ϕ preserves incidences and orientations,

$$\xi_1(\partial, \hat{g}) = \xi_2(\phi(\partial), \phi(\hat{g})).$$

Therefore, $\mathcal{B}_{N_1} \cong \mathcal{B}_{N_2}$, and hence, the two neutrosophic bidirected graphs are isomorphic. $\square$

Theorem 3.8. *Let $\mathcal{B}_N = (\top, l, F, \top_E, l_E, F_E, \xi)$ be a neutrosophic bidirected graph. If $V' \subseteq V$ and $E' = \{\{\zeta, \partial\} \in E : \zeta, \partial \in V'\} \subseteq E$. Then, the restriction $\mathcal{B}'_N = (\top', l', F', \top'_E, l'_E, F'_E, \xi')$ forms a neutrosophic bidirected subgraph of $\mathcal{B}_N$.*

Proof. Since $V' \subseteq V$ and $E' \subseteq E$, all vertex and edge neutrosophic values are inherited directly from $\mathcal{B}_N$. For every vertex $\partial \in V'$, because $\mathcal{B}_N$ is a neutrosophic bidirected graph,

$$0 \leq \top(\partial) + l(\partial) + F(\partial) \leq 3.$$

Hence,

$$0 \leq \top'(\partial) + l'(\partial) + F'(\partial) \leq 3.$$

Similarly, for every edge $\hat{g} \in E'$, we have

$$0 \leq \top_E(\hat{g}) + l_E(\hat{g}) + F_E(\hat{g}) \leq 3,$$

which implies

$$0 \leq \top'_E(\hat{g}) + l'_E(\hat{g}) + F'_E(\hat{g}) \leq 3.$$

Now, let

$$\hat{g} = \{\zeta, \partial\} \in E'.$$

Since $\hat{g} \in E$, the membership consistency conditions in $\mathcal{B}_N$ yield

$$\top_E(\hat{g}) \leq \min\{\top(\zeta), \top(\partial)\},$$

$$l_E(\hat{g}) \leq \max\{l(\zeta), l(\partial)\},$$

and

$$F_E(\hat{g}) \geq \max\{F(\zeta), F(\partial)\}.$$

Using the definitions of the restricted functions,

$$\top'_E(\hat{g}) \leq \min\{\top'(\zeta), \top'(\partial)\},$$

$$l'_E(\hat{g}) \leq \max\{l'(\zeta), l'(\partial)\},$$

and

$$F'_E(\hat{g}) \geq \max\{F'(\zeta), F'(\partial)\}.$$

Furthermore, since ξ' is the restriction of ξ ,

$$\xi'(\partial, \hat{g}) = 0 \iff \partial \notin \hat{g},$$

and for every

$$\hat{g} = \{\zeta, \partial\} \in E',$$

$$\xi'(\zeta, \hat{g}), \xi'(\partial, \hat{g}) \in \{-1, 1\}.$$

Therefore, $\mathcal{B}'_N$ satisfies all axioms of a neutrosophic bidirected graph and hence forms a neutrosophic bidirected subgraph of $\mathcal{B}_N$. $\square$

3.1. Structural Properties of Strong Neutrosophic Bidirected Graphs.

Theorem 3.9. *Let $\mathcal{B}_N = (\top, \text{I}, \text{F}, \top_E, \text{I}_E, \text{F}_E, \xi)$ be a strong neutrosophic bidirected graph. If $\mathcal{B}'_N$ is an induced neutrosophic bidirected subgraph of $\mathcal{B}_N$, then $\mathcal{B}'_N$ is also a strong neutrosophic bidirected graph.*

Proof. Let

$$V' \subseteq V$$

and let

$$E' = \{\{\zeta, \partial\} \in E : \zeta, \partial \in V'\}$$

be the edge set of the induced subgraph. Since $\mathcal{B}'_N$ is induced, all vertex and edge neutrosophic values are inherited from $\mathcal{B}_N$. Take any edge

$$\hat{g} = \{\zeta, \partial\} \in E'.$$

Since $\mathcal{B}_N$ is strong,

$$\top_E(\hat{g}) = \min\{\top(\zeta), \top(\partial)\}, \text{I}_E(\hat{g}) = \max\{\text{I}(\zeta), \text{I}(\partial)\}, \text{ and } \text{F}_E(\hat{g}) = \max\{\text{F}(\zeta), \text{F}(\partial)\}.$$

Because the induced subgraph preserves these values,

$$\top'_E(\hat{g}) = \min\{\top'(\zeta), \top'(\partial)\}, \text{I}'_E(\hat{g}) = \max\{\text{I}'(\zeta), \text{I}'(\partial)\}, \text{ and } \text{F}'_E(\hat{g}) = \max\{\text{F}'(\zeta), \text{F}'(\partial)\}.$$

Hence, every edge of $\mathcal{B}'_N$ satisfies the defining equalities of a strong neutrosophic bidirected graph. Therefore, $\mathcal{B}'_N$ is strong. $\square$

Theorem 3.10. *Let $\mathcal{B}_{N_1} \cong \mathcal{B}_{N_2}$. Then, $\mathcal{B}_{N_1}$ is strong if and only if $\mathcal{B}_{N_2}$ is strong.*

Proof. Let $\phi : V_1 \rightarrow V_2$ be an isomorphism between $\mathcal{B}_{N_1}$ and $\mathcal{B}_{N_2}$.

Assume that $\mathcal{B}_{N_1}$ is strong.

Take any edge

$$\{\phi(\zeta), \phi(\partial)\} \in E_2.$$

Since ϕ is an isomorphism,

$$\{\zeta, \partial\} \in E_1$$

and all neutrosophic values are preserved. Thus,

$$\top_2(\phi(\zeta)) = \top_1(\zeta), \quad \top_2(\phi(\partial)) = \top_1(\partial),$$

and similarly for the indeterminacy and falsity memberships. Since $\mathcal{B}_{N_1}$ is strong,

$$\top_{E_1}(\{\zeta, \partial\}) = \min\{\top_1(\zeta), \top_1(\partial)\}.$$

Applying ϕ ,

$$\top_{E_2}(\{\phi(\zeta), \phi(\partial)\}) = \min\{\top_2(\phi(\zeta)), \top_2(\phi(\partial))\}.$$

Similarly,

$$\text{I}_{E_2} = \max\{\text{I}_2(\phi(\zeta)), \text{I}_2(\phi(\partial))\},$$

and

$$\text{F}_{E_2} = \max\{\text{F}_2(\phi(\zeta)), \text{F}_2(\phi(\partial))\}.$$

Hence, $\mathcal{B}_{N_2}$ is strong.

The converse follows by applying the same argument to ϕ^{-1} . Therefore, strongness is preserved under isomorphism. $\square$

Theorem 3.11. *Let*

$$\mathcal{B}_N = (\top, \text{I}, \text{F}, \top_E, \text{I}_E, \text{F}_E, \xi)$$

be a neutrosophic bidirected graph. Define

$$\tilde{\top}_E(\{\zeta, \partial\}) = \min\{\top(\zeta), \top(\partial)\},$$

$$\tilde{\text{I}}_E(\{\zeta, \partial\}) = \max\{\text{I}(\zeta), \text{I}(\partial)\},$$

and

$$\tilde{F}_E(\{\zeta, \partial\}) = \max\{F(\zeta), F(\partial)\}.$$

Then

$$\tilde{\mathcal{B}}_N = (\top, \text{I}, F, \tilde{\top}_E, \tilde{\text{I}}_E, \tilde{F}_E, \xi)$$

is a strong neutrosophic bidirected graph.

Proof. For every edge

$$\hat{g} = \{\zeta, \partial\} \in E,$$

the definitions immediately give

$$\tilde{\top}_E(\hat{g}) = \min\{\top(\zeta), \top(\partial)\},$$

$$\tilde{\text{I}}_E(\hat{g}) = \max\{\text{I}(\zeta), \text{I}(\partial)\},$$

and

$$\tilde{F}_E(\hat{g}) = \max\{F(\zeta), F(\partial)\}.$$

Hence, the membership consistency conditions hold with equality. Moreover,

$$0 \leq \tilde{\top}_E, \tilde{\text{I}}_E, \tilde{F}_E \leq 1,$$

because minima and maxima of numbers in $[0, 1]$ remain in $[0, 1]$.

The bidirection function ξ is unchanged, so the incidence and local orientation conditions remain valid. Therefore

$$\tilde{\mathcal{B}}_N$$

satisfies all requirements of a neutrosophic bidirected graph and, by construction, satisfies the defining equalities of a strong neutrosophic bidirected graph.

Hence,

$$\tilde{\mathcal{B}}_N$$

is strong. □

Example 3.3. Consider the neutrosophic bidirected graph

$$\mathcal{B}_N = (\top, \text{I}, F, \top_E, \text{I}_E, F_E, \xi)$$

on the crisp graph

$$G^* = (V, E), \quad V = \{v_1, v_2\}, \quad E = \{\hat{g}\},$$

where

$$\hat{g} = \{v_1, v_2\}.$$

Let

$$(\top(v_1), \text{I}(v_1), F(v_1)) = (0.9, 0.1, 0.2),$$

$$(\top(v_2), \text{I}(v_2), F(v_2)) = (0.7, 0.3, 0.4).$$

Assume that

$$(\top_E(\hat{g}), \text{I}_E(\hat{g}), F_E(\hat{g})) = (0.5, 0.2, 0.4).$$

Since

$$\min\{0.9, 0.7\} = 0.7 \neq 0.5,$$

the graph is not strong.

Now define

$$\tilde{\top}_E(\hat{g}) = \min\{0.9, 0.7\} = 0.7,$$

$$\tilde{\text{I}}_E(\hat{g}) = \max\{0.1, 0.3\} = 0.3,$$

and

$$\tilde{F}_E(\hat{g}) = \max\{0.2, 0.4\} = 0.4.$$

Hence

$$(\tilde{\top}_E, \tilde{\text{I}}_E, \tilde{F}_E) = (0.7, 0.3, 0.4).$$

Therefore

$$\tilde{\mathcal{B}}_N = (\top, \mathsf{l}, \mathsf{F}, \tilde{\top}_E, \tilde{\mathsf{l}}_E, \tilde{\mathsf{F}}_E, \xi)$$

satisfies

$$\begin{aligned}\tilde{\top}_E(\hat{\mathbf{g}}) &= \min\{\top(v_1), \top(v_2)\}, \\ \tilde{\mathsf{l}}_E(\hat{\mathbf{g}}) &= \max\{\mathsf{l}(v_1), \mathsf{l}(v_2)\},\end{aligned}$$

and

$$\tilde{\mathsf{F}}_E(\hat{\mathbf{g}}) = \max\{\mathsf{F}(v_1), \mathsf{F}(v_2)\}.$$

Thus

$$\tilde{\mathcal{B}}_N$$

is a strong neutrosophic bidirected graph.

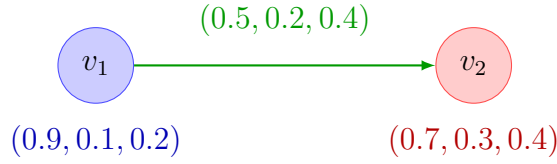


FIGURE 5. The original neutrosophic bidirected graph $\mathcal{B}_N$, which is not strong.

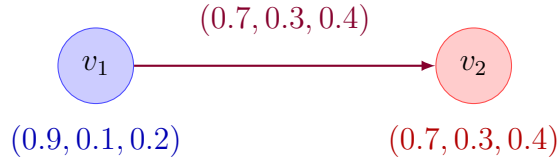


FIGURE 6. The canonical strong extension $\tilde{\mathcal{B}}_N$.

Remark 3.1. Observe that the vertex neutrosophic values and the bidirection function remain unchanged. Only the edge neutrosophic values are replaced by their extremal admissible values, thereby converting $\mathcal{B}_N$ into a strong neutrosophic bidirected graph.

Theorem 3.12. Let $\mathcal{B}_N = (\top, \mathsf{l}, \mathsf{F}, \top_E, \mathsf{l}_E, \mathsf{F}_E, \xi)$ be a neutrosophic bidirected graph.

Define $\xi^-(\partial, \hat{\mathbf{g}}) = -\xi(\partial, \hat{\mathbf{g}})$ for every $\partial \in V$, $\hat{\mathbf{g}} \in E$. Then, $\mathcal{B}_N^- = (\top, \mathsf{l}, \mathsf{F}, \top_E, \mathsf{l}_E, \mathsf{F}_E, \xi^-)$ is also a neutrosophic bidirected graph.

Proof. Observe that the orientation reversal changes only the bidirection function ξ . The vertex memberships

$$\top, \quad \mathsf{l}, \quad \mathsf{F},$$

and the edge memberships

$$\top_E, \quad \mathsf{l}_E, \quad \mathsf{F}_E$$

remain unchanged.

Therefore, the vertex neutrosophic constraint, the edge neutrosophic constraint, and the membership consistency conditions continue to hold for $\mathcal{B}_N^-$.

It remains to verify the incidence and local orientation conditions.

Let $\partial \in V$ and $\hat{\mathbf{g}} \in E$. Then

$$\xi^-(\partial, \hat{\mathbf{g}}) = 0 \iff -\xi(\partial, \hat{\mathbf{g}}) = 0 \iff \xi(\partial, \hat{\mathbf{g}}) = 0.$$

Since $\mathcal{B}_N$ is a neutrosophic bidirected graph,

$$\xi(\partial, \hat{\mathbf{g}}) = 0 \iff \partial \notin \hat{\mathbf{g}}.$$

Hence

$$\xi^-(\partial, \hat{\mathbf{g}}) = 0 \iff \partial \notin \hat{\mathbf{g}},$$

so the incidence condition is preserved.

Now let

$$\hat{\mathbf{g}} = \{\zeta, \partial\} \in E.$$

Because $\mathcal{B}_N$ satisfies the local orientation condition,

$$\xi(\zeta, \hat{\mathbf{g}}), \xi(\partial, \hat{\mathbf{g}}) \in \{-1, 1\}.$$

Multiplying by -1 yields

$$\xi^-(\zeta, \hat{\mathbf{g}}), \xi^-(\partial, \hat{\mathbf{g}}) \in \{-1, 1\}.$$

Thus the local orientation condition also remains valid.

Consequently, all defining axioms of a neutrosophic bidirected graph are satisfied by

$$\mathcal{B}_N^-.$$

Therefore,

$$\mathcal{B}_N^-$$

is a neutrosophic bidirected graph. □

Example 3.4. Consider the neutrosophic bidirected graph

$$\mathcal{B}_N = (\top, \mathbf{l}, \mathbf{F}, \top_E, \mathbf{l}_E, \mathbf{F}_E, \xi)$$

on the crisp graph

$$G^* = (V, E), \quad V = \{v_1, v_2\}, \quad E = \{\hat{\mathbf{g}}\},$$

where

$$\hat{\mathbf{g}} = \{v_1, v_2\}.$$

Assign the vertex neutrosophic values

$$(\top(v_1), \mathbf{l}(v_1), \mathbf{F}(v_1)) = (0.8, 0.2, 0.1),$$

$$(\top(v_2), \mathbf{l}(v_2), \mathbf{F}(v_2)) = (0.6, 0.3, 0.2).$$

For the edge $\hat{\mathbf{g}}$, let

$$(\top_E(\hat{\mathbf{g}}), \mathbf{l}_E(\hat{\mathbf{g}}), \mathbf{F}_E(\hat{\mathbf{g}})) = (0.6, 0.3, 0.2).$$

Define the bidirection function by

$$\xi(v_1, \hat{\mathbf{g}}) = 1, \quad \xi(v_2, \hat{\mathbf{g}}) = -1.$$

Hence the edge is oriented from v_1 toward v_2 .

Now define

$$\xi^-(v_i, \hat{\mathbf{g}}) = -\xi(v_i, \hat{\mathbf{g}}), \quad i = 1, 2.$$

Then

$$\xi^-(v_1, \hat{\mathbf{g}}) = -1, \quad \xi^-(v_2, \hat{\mathbf{g}}) = 1.$$

Thus the orientation of the edge is reversed while all neutrosophic vertex and edge values remain unchanged. Therefore

$$\mathcal{B}_N^- = (\top, \mathbf{l}, \mathbf{F}, \top_E, \mathbf{l}_E, \mathbf{F}_E, \xi^-)$$

is again a neutrosophic bidirected graph.

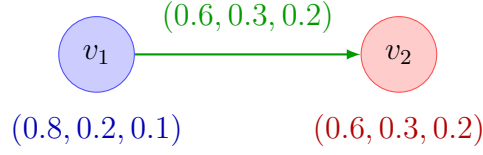


FIGURE 7. The neutrosophic bidirected graph $\mathcal{B}_N$.

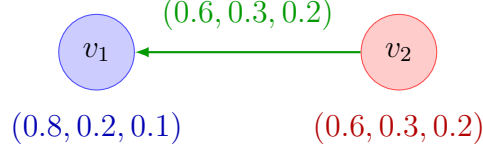


FIGURE 8. The orientation-reversed neutrosophic bidirected graph $\mathcal{B}_N^-$.

4. APPLICATIONS OF NEUTROSOPHIC BIDIRECTED GRAPHS

The theory of neutrosophic bidirected graphs provides a generalized and flexible mathematical framework for studying complex systems involving simultaneous uncertainty, indeterminacy, inconsistency, and directional interactions. Since the proposed structure extends fuzzy, intuitionistic fuzzy, and classical bidirected graph models, it possesses significant potential in both theoretical investigations and real-world applications.

Let $G = (V, E, \sigma, \mu, \eta, \phi, \beta)$ be a neutrosophic bidirected graph, where $\sigma = (T_\sigma, I_\sigma, F_\sigma)$ and $\mu = (T_\mu, I_\mu, F_\mu)$ denote the neutrosophic vertex and edge functions, respectively, and $\beta : E \rightarrow \{(-, -), (-, +), (+, -), (+, +)\}$ is the bidirection mapping.

The following applications illustrate the usefulness of this model.

4.1. Communication Networks. Consider a communication network in which each vertex represents a communication device and each edge represents a communication channel. For an edge $uv \in E$, $T_\mu(uv)$ may denote the reliability of successful transmission, $I_\mu(uv)$ the uncertainty caused by signal fluctuation, and $F_\mu(uv)$ the probability of communication failure.

The bidirection

$$\beta(uv) = (+, -)$$

indicates that the transmission from u to v is outgoing at u and incoming at v . Hence, neutrosophic bidirected graphs can model uncertain bidirectional communication systems more effectively than ordinary directed graphs.

4.2. Transportation Systems. Let vertices represent cities and edges represent roads connecting them. Suppose $uv \in E$. Then:

$$T_\mu(uv) = \text{degree of road availability,}$$

$$I_\mu(uv) = \text{uncertainty due to weather or traffic,}$$

and

$$F_\mu(uv) = \text{degree of blockage or failure.}$$

If $\beta(uv) = (+, +)$, then, the road allows transportation in both directions. Thus, neutrosophic bidirected graphs provide a mathematical tool for studying transportation systems under uncertain conditions.

4.3. Social Networks. In social networks, vertices correspond to individuals and edges represent relationships between them. For $uv \in E$, $T_\mu(uv)$ may represent the strength of friendship, $I_\mu(uv)$ the hesitation or ambiguity in the relationship, and $F_\mu(uv)$ the degree of conflict.

The bidirection function distinguishes mutual and non-mutual interactions. Therefore, the model is applicable in trust analysis, opinion dynamics, and recommendation systems.

4.4. Decision-Making Problems. Suppose $A = \{a_1, a_2, \dots, a_n\}$ is a collection of alternatives in a decision-making problem. A neutrosophic bidirected graph may be constructed by assigning each alternative to a vertex and defining edges according to preference relations.

For an edge $a_i a_j$, $T_\mu(a_i a_j)$ denotes the degree to which a_i is preferred over a_j , $I_\mu(a_i a_j)$ represents indeterminate assessment, and $F_\mu(a_i a_j)$ indicates rejection or opposition.

Hence, neutrosophic bidirected graphs are useful in multicriteria decision-making under uncertainty.

4.5. Biological Networks. In biological systems, vertices may represent neurons, proteins, or genes, while edges describe interactions among them. For example, $T_\mu(uv)$ can measure activation strength, $I_\mu(uv)$ the uncertainty in biological interaction, and $F_\mu(uv)$ the inhibitory effect.

The bidirection mapping models mutual biological interactions. Consequently, neutrosophic bidirected graphs are applicable in bioinformatics and neural network analysis.

4.6. Electrical and Power Networks. Let vertices denote substations and edges denote transmission lines. For an edge uv , $T_\mu(uv)$ represents operational efficiency, $I_\mu(uv)$ represents uncertain load variation, and $F_\mu(uv)$ represents the degree of line failure.

The bidirection mapping characterizes two-way power flow in smart grids. Therefore, neutrosophic bidirected graphs can be used in fault diagnosis and power distribution analysis.

To illustrate the proposed concept of neutrosophic bidirected graphs and the role of neutrosophic membership values together with bidirections, we present the following example. The example demonstrates how truth, indeterminacy, and falsity memberships can be assigned to vertices and edges, while bidirections describe the directional behavior of the connections in an uncertain communication network.

Example 4.1. Consider the neutrosophic bidirected graph $\mathcal{G} = (\mathcal{V}, \mathbf{E}, \sigma, \mu, \beta)$, where $V = \{\partial_1, \partial_2, \partial_3\}$ and $\mathbf{E} = \{\partial_1 \partial_2, \partial_2 \partial_3, \partial_1 \partial_3\}$. Define the neutrosophic vertex membership functions by

$$\sigma(\partial_1) = (0.8, 0.1, 0.2), \quad \sigma(\partial_2) = (0.7, 0.2, 0.3), \quad \text{and} \quad \sigma(\partial_3) = (0.9, 0.1, 0.1).$$

Further, define the neutrosophic edge membership functions as

$$\mu(\partial_1 \partial_2) = (0.6, 0.2, 0.3), \quad \mu(\partial_2 \partial_3) = (0.5, 0.3, 0.4), \quad \text{and} \quad \mu(\partial_1 \partial_3) = (0.7, 0.1, 0.2).$$

The bidirection mapping $\beta : \mathbf{E} \rightarrow \{(+, +), (+, -), (-, +), (-, -)\}$ is defined by

$$\beta(\partial_1 \partial_2) = (+, -), \quad \beta(\partial_2 \partial_3) = (-, +), \quad \text{and} \quad \beta(\partial_1 \partial_3) = (+, +).$$

This graph may represent a communication network in which:

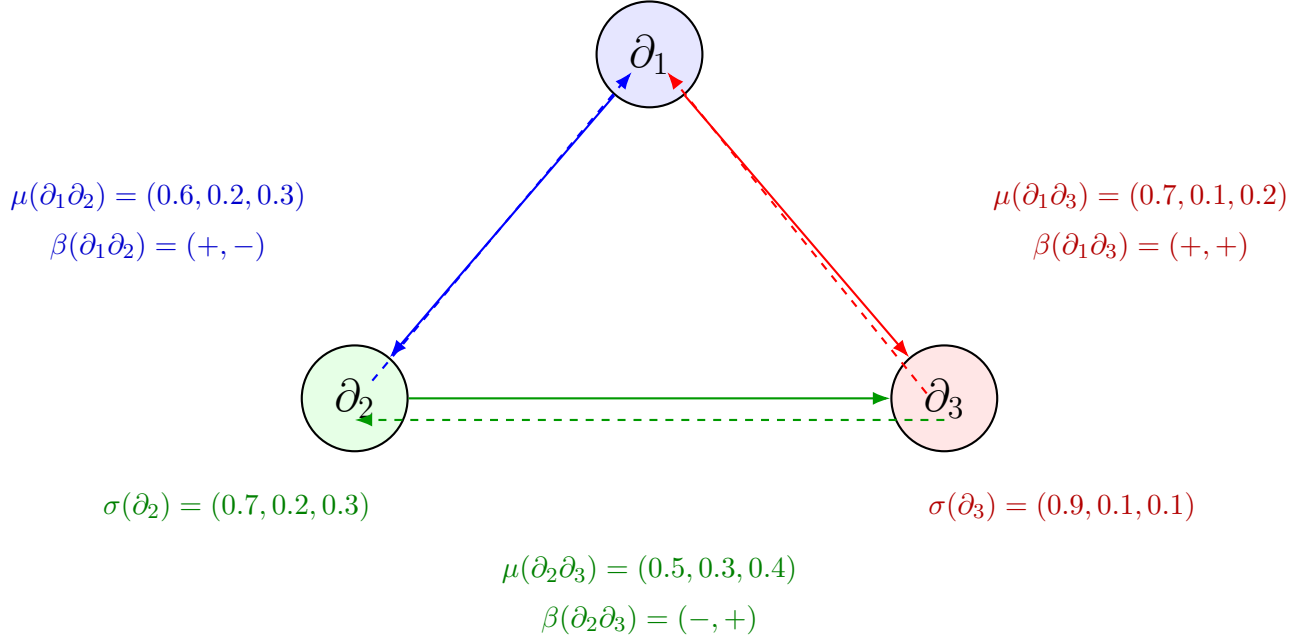
- (1) vertices correspond to communication devices,
- (2) edges represent communication links,
- (3) the first component of each neutrosophic value denotes the degree of reliable communication,
- (4) the second component denotes indeterminacy caused by unstable signals, and
- (5) the third component denotes the degree of transmission failure.

For example, $\mu(\partial_1 \partial_2) = (0.6, 0.2, 0.3)$ indicates that the communication link between ∂_1 and ∂_2 has reliability degree 0.6, indeterminacy degree 0.2, and failure degree 0.3. Moreover, $\beta(\partial_1 \partial_2) = (+, -)$ shows that the edge behaves as outgoing at ∂_1 and incoming at ∂_2 , while $\beta(\partial_1 \partial_3) = (+, +)$ represents a bidirectional interaction at both incident vertices.

Neutrosophic Bidirected Graph

$$\mathcal{G} = (\mathcal{V}, E, \sigma, \mu, \beta)$$

$$\sigma(\partial_1) = (0.8, 0.1, 0.2)$$



Legend (Bidirection β)	Information
$(+, +)$: bidirectional at both ends	• Vertices represent communication devices
$(+, -)$: outgoing at first, incoming at second	• Edges represent communication links
$(-, +)$: incoming at first, outgoing at second	• $(\top, \text{I}, \text{F})$ = truth, indeterminacy, falsity

FIGURE 9. Neutrosophic bidirected graph associated with Example 1.

Hence, G is a neutrosophic bidirected graph modeling an uncertain and partially indeterminate communication system.

5. CONCLUSION

In this paper, we introduced the concept of neutrosophic bidirected graphs as a natural extension of intuitionistic fuzzy bidirected graphs by incorporating truth-membership, indeterminacy-membership, and falsity-membership functions on vertices and edges together with a bidirection mapping. Several fundamental properties and structural characteristics of neutrosophic bidirected graphs were established.

We investigated reductions of neutrosophic bidirected graphs to fuzzy bidirected graphs, crisp bidirected graphs, and neutrosophic directed graphs under suitable conditions. Furthermore, operations such as intersection and union were studied, and important concepts including strong neutrosophic bidirected graphs, homomorphisms, and isomorphisms were analyzed in detail. The obtained results demonstrate that neutrosophic bidirected graphs provide a flexible and generalized framework for modeling uncertainty, indeterminacy, and inconsistency in bidirectional network structures.

The proposed structure may serve as a basis for future investigations in neutrosophic graph theory. Possible future directions include the study of neutrosophic connectivity, regularity, coloring, domination, spectral properties, and applications to optimization, decision-making, artificial intelligence, and complex network analysis.

REFERENCES

- [1] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, **8**(3), 338–353.
- [2] Rosenfeld, A. (1975). Fuzzy graphs. In L. A. Zadeh, K. S. Fu, M. Shimura, & A. Kaufmann (Eds.), *Fuzzy Sets and Their Applications* (pp. 77–95). Academic Press.
- [3] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, **20**(1), 87–96.
- [4] Parvathi, R., Karunambigai, M. G., & Atanassov, K. T. (2006). Operations on intuitionistic fuzzy graphs. In *Proceedings of IEEE International Conference on Fuzzy Systems*, 1396–1401.
- [5] Akram, M., & Dudek, W. A. (2011). Intuitionistic fuzzy hypergraphs with applications. *Information Sciences*, **218**, 182–193.
- [6] Bhutani, K. R., & Rosenfeld, A. (2003). Strong arcs in fuzzy graphs. *Information Sciences*, **152**, 319–322.
- [7] Smarandache, F. (1999). *Neutrosophy: Neutrosophic Probability, Set, and Logic*. American Research Press.
- [8] Smarandache, F., & Broumi, S. (Eds.). (2019). *Neutrosophic graph theory and algorithms*. IGI Global.
- [9] Broumi, S., Talea, M., Bakali, A., & Smarandache, F. (2016). Single valued neutrosophic graphs. *Journal of New Theory*, **10**, 86–101.
- [10] Broumi, S., Sundareswaran, R., Shanmugapriya, M., Bakali, A., & Talea, M. (2022). Theory and applications of fermatean neutrosophic graphs. *Neutrosophic sets and systems*, **50**(2), 248–286.
- [11] Raut, P. K., Pramanik, S., & S Mohanty, B. (2025). An overview of Fermatean neutrosophic graphs. *Neutrosophic Sets and Systems*, **82**(1), 40.
- [12] Ye, J. (2014). Single-valued neutrosophic minimum spanning tree and its clustering method. *Journal of intelligent Systems*, **23**(3), 311–324.
- [13] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman, “Single Valued Neutrosophic Sets,” *Multi-space and Multistructure*, vol. 4, pp. 410–413, 2010.